**Formulas**

$F = K_e \frac{|q_1||q_2|}{r^2}$ (Magnitude of Coulomb force between two point charges.)

$\vec{\mathbf{F}}_{on1} = \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{14} + \ldots$ (Superposition principle for point charges)

$E = K_e \frac{|q|}{r^2}$ (Electric field of a point charge)

$\vec{\mathbf{F}} = q\vec{\mathbf{E}}$ (Relationship between electric field and Coulomb force)

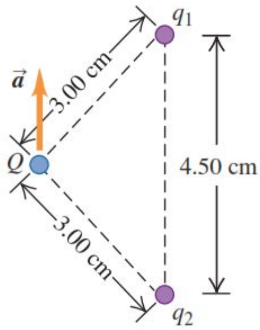
$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 + \ldots$ (Superposition principle for E. fields)

**Solving continuous distributions:**

1. Take a differential of the charge $dq$
2. Find $dE$ (magnitude) in terms of $q$
3. Find $dE_x$ and $dE_y$ (magnitudes) and set their directions beforehand
4. Integrate $dE_x$ to get $E_x$ and $dE_y$ to get $E_y$
5. Combine the magnitude from step 4 and direction from step 3 to get a full expression for $\vec{\mathbf{E}}$ in vector form.

## Problem 1

**University Physics, problem 76. Exam Problem.**
Two point charges $q_1$ and $q_2$ are held in place 4.5 cm apart. Another charge Q = -1.75 $\mu$C of mass 5.0 g is initially located 3.0 cm from each of these charges and released from rest. You observe that the initial acceleration of $Q$ is 324.0 m/s$^2$ upward, parallel to the line connecting the two point charges. Find $q_1$ and $q_2$.




### Given

$Q = -1.74\ \mu$C

$m_Q = 5.00$ g

$a = 324$ m/s$^2$

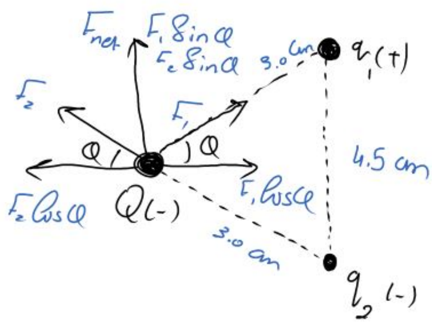
### Solution

The charge $Q$ is accelerated upward, which means that it is attracted to $q_1$ and repelled by $q_2$. Therefore $q_1 > 0$ and $q_2 < 0$. Also, since the acceleration (and thus the net force) is directed upwards, with no x-component, then it must be that $F_{Qq_1} = F_{Qq_2}$, and thus $|q_1| = |q_2|$. Therefore, we have: $q_1 = -q_2$. This can be proved mathematically in an optional step as demonstrated below.

$$\vec{\mathbf{F}}_{Qq_1} cos\theta + \vec{\mathbf{F}}_{Qq_2} cos\theta = 0 \implies F_{Qq_1} = F_{Qq_1}$$
$$k\frac{|Q||q_1|}{r^2} = k\frac{|Q||q_2|}{r^2} \implies |q_1| = |q_2|$$

But $q_1 > 0$ and $q_2 < 0$ therefore:

$$q_1 = -q_2$$



$$\vec{F} = \vec{F}_{Qq_1} sin\theta + \vec{F}_{Qq_2} sin\theta, \quad \vec{F} = m\vec{a}$$
$$a = \frac{1}{m} sin\theta \left( k\frac{|Q||q_1|}{r^2} + k\frac{|Q||q_2|}{r^2} \right)$$
$$a = \frac{k|Q|sin\theta}{mr^2} \left(|q_1| + |q_1|\right)$$
$$324 = \frac{\left(9 \times 10^9\right)\left(|-1.75 \times 10^{-6}|\right)\left(\frac{2.25}{3}\right)}{\left(5 \times 10^{-3}\right)\left(3 \times 10^{-2}\right)^2} \left(2|q_1|\right)$$

$$\boxed{q_1 = 6.17 \times 10^{-8} C}$$
$$\boxed{q_2 = -q_1 = -6.17 \times 10^{-8} C}$$

## Problem 2

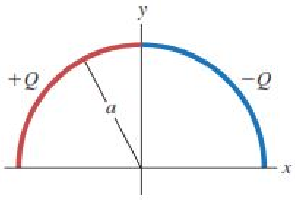
### Exam Problem

a. In the given figure, the radius of the semicircle is $R$. The charge on the right half is $-Q$ and $+Q$ on the left half. Find the electric field (magnitude and direction) at the center of the semicircle.

b. Assume that the charge distribution is no longer uniform but has a linear charge density $\lambda(\theta) = \frac{Q}{\pi R}(1 + \sin\theta\cos\theta)$. Find the electric field (magnitude and direction) at the center of the semicircle.

### Solution

### Part (a)




When solving electric field or Coulomb force problems, we always completely separate the magnitude and direction. First set the direction, then find the magnitude.
**Direction:** For the right half, the field components will be in the +x direction and +y direction. For the left half, the field components will be in the +x direction and the -y direction. We can thus cancel the y-component due to symmetry.
**Magnitude:** The magnitude of the fields for both halves will be:

$$dE = K_e \frac{dQ}{R^2}$$
$$\lambda = \frac{dQ}{dl} = \frac{dQ}{Rd\theta} = \frac{Q}{\pi R/2} \implies dQ = \frac{2d\theta Q}{\pi}$$
$$dE = K_e \frac{2d\theta Q}{\pi R^2}$$
$$dE_x = K_e \frac{2d\theta Q}{\pi R^2}\cos\theta$$
$$E_{x_1} = E_{x_2} = K_e \frac{2Q}{\pi R^2}\int_0^{\pi/2}\cos\theta d\theta$$
$$E_{x_1} = E_{x_2} = K_e \frac{2Q}{\pi R^2}\sin\theta\Big|_0^{\pi/2} = K_e \frac{2Q}{\pi R^2}$$
$$E_x = E_{x_1} + E_{x_2} = 2K_e \frac{2Q}{\pi R^2} = \boxed{4K_e \frac{Q}{\pi R^2}}$$

### Part (b)

This type of charge distribution is dependent on the angle $\theta$, which means that the charge on the semicircle increases as the angle increases. Note that it is not uniform, which means that $\lambda \neq \frac{Q}{L}$
**Direction:** From $\theta = 0$ to $\theta = \pi/2$, the x-component field is in the -x direction, and in the +x direction from $\theta = \pi/2$ to $\theta = \pi$. However, the x-component of the field caused by the first half of the semicircle is less than the x-component of the field caused by the second half, thus the resulting field is in the +x direction.
As for the y-direction, the field is in the -y direction.

**Magnitude:** The magnitude of the field is as follows:

$$dE = K_e \frac{dQ}{R^2}$$
$$\lambda = \frac{dQ}{dl} = \frac{dQ}{Rd\theta} = \frac{Q}{\pi R}\left(1+\sin\theta\cos\theta\right) \implies dQ = \frac{Q}{\pi}\left(1+\sin\theta\cos\theta\right) d\theta$$
$$dE = K_e \frac{Q}{\pi R^2}\left(1+\sin\theta\cos\theta\right) d\theta$$
$$dE_x = K_e \frac{Q}{\pi R^2}\left(1+\sin\theta\cos\theta\right)\cos\theta d\theta = K_e \frac{Q}{\pi R^2}\left(\cos\theta + \sin\theta\cos^2\theta\right) d\theta$$
$$E_x = K_e \frac{Q}{\pi R^2}\int_0^{\pi}\left(\cos\theta + \sin\theta\cos^2\theta\right) d\theta\,;u=\cos\theta\,,du=-\sin\theta d\theta\,;u\left(0\right)=1;u\left(\pi\right)=-1;$$
$$E_x = K_e \frac{Q}{\pi R^2}\left(\int_0^{\pi}\cos\theta d\theta - \int_1^{-1}u^2du\right) = K_e \frac{Q}{\pi R^2}\left(\sin\theta\Big|_0^{\pi} - \frac{u^3}{3}\Big|_1^{-1}\right) = K_e \frac{Q}{\pi R^2}\left(\frac{1}{3}+\frac{1}{3}\right)$$
$$E_x = \frac{2QK_e}{3\pi R^2} \qquad \text{in the +x direction}$$
$$dE_y = K_e \frac{Q}{\pi R^2}\left(1+\sin\theta\cos\theta\right)\sin\theta d\theta = K_e \frac{Q}{\pi R^2}\left(\sin\theta + \sin^2\theta\cos\theta\right) d\theta$$
$$E_y = K_e \frac{Q}{\pi R^2}\int_0^{\pi}\left(\cos\theta + \sin^2\theta\cos\theta\right) d\theta\,;u=\sin\theta\,,du=\cos\theta d\theta\, u\left(0\right)=0;u\left(\pi\right)=0;$$
$$E_y = K_e \frac{Q}{\pi R^2}\left(\int_0^{\pi}\sin\theta d\theta - \int_0^{0}u^2du\right) = K_e \frac{Q}{\pi R^2}\left(-\cos\theta\Big|_0^{\pi}\right) = K_e \frac{Q}{\pi R^2}\left(-\left(-1-1\right)\right)$$
$$E_y = \frac{2QK_e}{\pi R^2} \qquad \text{in the -y direction}$$
$$\vec{\mathbf{E}} = \frac{2QK_e}{\pi R^2}\left(\frac{1}{3}\hat{\imath} - \hat{\jmath}\right)$$
$$E = \frac{2QK_e}{\pi R^2}\sqrt{\left(\frac{1}{3}\right)^2 + 1^2} = \boxed{\frac{2\sqrt{10}}{3}\cdot\frac{QK_e}{\pi R^2}}$$
$$\tan\phi = -3 \implies \boxed{\phi \approx 72^o \text{ below the +x axis}}$$