Relevant Formulas

$Z = R + jX$ (complex impedance)

$X_L = j\omega L$ (reactance of an inductor)

$X_C = \frac{-j}{\omega C}$ (reactance of a capacitor)

$|Z| = \sqrt{\Re(Z)^2 + \Im(Z)^2}$ (magnitude of impedance)

$Z_{eq} = Z_1 + Z_2 + Z_3 + \dots$ (series connection)

$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \dots$ (parallel connection)

$\tilde{V} = V_0 e^{j\omega t}$ (voltage phasor)

$\tilde{I} = I_0 e^{j(\omega t - \phi)} = \frac{\tilde{V}}{Z}$ (current phasor)

$I_0 = \frac{V_0}{|Z|}$ (current amplitude)

$\tan\phi = \frac{\Im(Z)}{\Re(Z)}$ (tangent of phase angle)

$\tilde{I}_C = \frac{\tilde{V}_C}{X_C}$ (capacitor current phasor)

$\tilde{I}_L = \frac{\tilde{V}_C}{X_L}$ (inductor current phasor)

$\phi = 0$ (resonance condition)

$P_{av} = \frac{I_0 V_0}{2}\cos\phi = I_{rms}V_{rms}\cos\phi$ (average power)

$P_{av,R} = I_{rms}^2 R$ (average power dissipated in resistor)

In simple (all parallel or all series) AC RLC Circuits, a helpful mnemonic is:

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In inductors, current lags voltage by $\pi/2$ and in capacitors, current leads voltage by $\pi/2$, and in resistors, in phase with the voltage
Therefore:

$\tilde{I}_L = I_{0,L}\, e^{j(\omega t - \frac{\pi}{2})}$

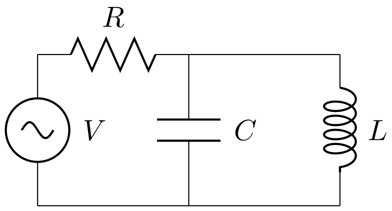
$\tilde{I}_C = I_{0,C}\, e^{j(\omega t + \frac{\pi}{2})}$

$\tilde{I}_R = I_{0,R}\, e^{j(\omega t)}$

Problem 1. RCL circuit

In the given circuit, calculate the current in each circuit element given that

$$V = V_0 sin(\omega t)$$



(a) Find the complex impedance Z.

(b) Find the current phasors in the 3 circuit elements.

Solution

Part (a)

We know that that

$$Z = R + jX$$

The impedances of the circuit elements are:

$$X_L = j\omega L$$

and

$$X_C = \frac{-j}{\omega C}$$

We first find the complex impedance of the parallel LC combination

$$1/Z_{LC} = 1/X_L + 1/X_C = \frac{-j}{\omega L} + j\omega C$$

$$1/Z_{LC} = j\frac{\omega^2 LC - 1}{\omega L}$$

$$Z_{XL} = j(\frac{\omega L}{1 - \omega^2 LC})$$

We can then find the complex impedance of the entire system:

$$Z = R + Z_{XL} = R + j(\frac{\omega L}{1 - \omega^2 LC})$$

$$\boxed{Z = R + j(\frac{\omega L}{1-\omega^2 LC})}$$

Knowing that:

$$|Z| = \sqrt{\Re(Z)^2 + \Im(Z)^2}$$

We can calculate the impedance of the circuit:

$$|Z| = \sqrt{R^2 + \frac{\omega^2 L^2}{(1 - \omega^2 LC)^2}}$$

Knowing that:

$$tan(\phi) = \frac{\Im(Z)}{\Re(Z)}$$

We can calculate the tangent of the phase angle ϕ:

$$tan(\phi) = \frac{\omega L}{R(1 - \omega^2 LC)}$$

Part (b)

We can represent the current in the circuit as a whole and the voltage of the source as phasors:

$$\tilde{V} = V_0 e^{j\omega t}$$

$$\tilde{I} = I_0 e^{j(\omega t - \phi)}$$

Where

$$I_0 = \frac{V_0}{|Z|}$$

Since R is in series with the LC combination, we know that

$$\tilde{I}_R = \tilde{I}_C + \tilde{I}_L = \tilde{I} = \frac{V_0}{|Z|} e^{j(wt-\phi)}$$

$$\boxed{\tilde{I}_R = \frac{V_0}{|Z|} e^{j(wt-\phi)}}$$

Since C and L are in parallel, they have the same voltage. Since they are in series with R

$$\tilde{V}_C = \tilde{V}_L = \tilde{V} - \tilde{V}_R$$

$$\tilde{V}_C = V_0 e^{jwt} - \frac{RV_0}{|Z|} e^{j(wt-\phi})$$

$$\tilde{V}_C = V_0 e^{jwt}(1 - \frac{R}{|Z|} e^{-j\phi})$$

We know that:

$$\tilde{I}_C = \frac{\tilde{V}_C}{X_C}$$

Therefore we can deduce that:

$$\tilde{I}_C = V_0 e^{jwt}(1 - \frac{R}{|Z|} e^{-j\phi})(j\omega C); \qquad j = e^{j\frac{\pi}{2}}$$

$$\boxed{\tilde{I}_C = V_0 \omega C e^{j(\omega t + \frac{\pi}{2})}(1 - \frac{R}{|Z|} e^{-j\phi})}$$

We know that:

$$\tilde{I}_L = \frac{\tilde{V}_C}{X_C}$$

Therefore we can deduce that:

$$\tilde{I}_L = V_0 e^{jwt}(1 - \frac{R}{|Z|} e^{-j\phi})(\frac{-j}{\omega L}); \qquad -j = e^{j\frac{-\pi}{2}}$$

$$\boxed{\tilde{I}_L = \frac{V_0}{\omega L} e^{j(\omega t - \frac{\pi}{2})}(1 - \frac{R}{|Z|} e^{-j\phi})}$$