Problem 1

University Physics, problem 67

A region in space contains a total positive charge that is distributed spherically such that the volume charge density $\rho(r)$ is given by

$$\rho(r) = 3\alpha r/(2R) \qquad \text{for } r \leq R/2$$
$$\rho(r) = \alpha[1-(r/R)^2] \qquad \text{for } R/2 \leq r \leq R$$
$$\rho(r) = 0 \qquad \text{for } r \geq R$$

Here α is a positive constant having units of C/m^3

a. Determine α in terms of Q and R

b. Using Gausss law, derive an expression for the magnitude of the electric field as a function of r. Do this separately for all three regions. Express your answers in terms of the total charge.

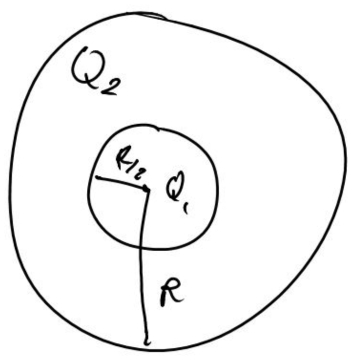
c. What fraction of the total charge is contained within the region $R/2 \leq r \leq R$

d. What is the magnitude of $\vec{E}$ at $r = R/2$

e. If an electron with charge $q' = -e$ is released from rest at any point in any of the three regions, the resulting motion will be oscillatory but not simple harmonic. Why?

Solution

Part (a)



$$\rho = \frac{dQ}{dV} \implies Q = \int_a^b \rho \, dV$$
$$dV = Adr = 4\pi r^2 dr$$

$$Q_1 = \int_0^{R/2} \left(\frac{3\alpha r}{2R}\right)\left(4\pi r^2\right) dr = \frac{6\pi\alpha}{R}\int_0^{R/2} r^3 dr = \frac{6\pi\alpha}{R}\frac{1}{4}\frac{R}{16} = \frac{3}{32}\pi\alpha R^3 \tag{1}$$

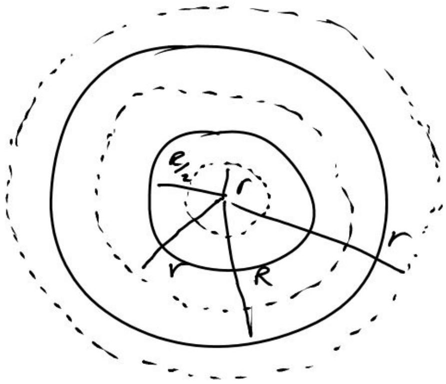
$$Q_2 = \int_{R/2}^{R} \alpha\left(1-\left(\frac{r}{R}\right)^2\right)\left(4\pi r^2\right) dr = 4\pi\alpha R^3\left(\frac{7}{24}-\frac{31}{160}\right) = \frac{47}{120}\pi\alpha R^3 \tag{2}$$

From (1) and (2)

$$Q = Q_1 + Q_2 = \frac{47}{120}\pi\alpha R^3 + \frac{3}{32}\pi\alpha R^3 = \frac{233}{480}\pi\alpha R^3$$

$$\boxed{\alpha = \frac{480Q}{233\pi R^3}}$$

Part (b)



For the region where $r \leq R/2$

$$\int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{en}}{\epsilon_0}, \text{ the enclosed charge is a fraction of } Q_1$$

$$Q_{en} = \int_0^r 3\alpha \frac{r}{2R}\left(4\pi r^2 dr\right) = \int_0^r 3\left(\frac{480Q}{233\pi R^3}\right)\left(\frac{r}{2R}\right)\left(4\pi r^2 dr\right) = \frac{720Q}{233R^4} \cdot 4\int_0^r r^3 dr = \frac{720Qr^4}{233R^4}$$

$$E_1 \cdot 4\pi r^2 = \frac{720Qr^4}{233R^4\epsilon_0}$$

$$\boxed{E_1 = \frac{180Qr^2}{233\pi\epsilon_0 R^4}}$$

For the region where $R/2 \leq r \leq R$

$$\int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{en}}{\epsilon_0}, \text{ the enclosed charge is the sum of } Q_1 \text{ and a fraction of } Q_2$$

$$Q_{en} = Q_1 + \int_{R/2}^r \alpha\left(1 - \left(\frac{r'}{R}\right)^2\right)\left(4\pi r'^2\right)dr' = Q_1 + 4\pi\alpha\int_{R/2}^r \left(r'^2 - \frac{r'^4}{R^2}\right)dr' = Q_1 + 4\pi\alpha\left(\frac{r'^3}{3} - \frac{r'^5}{5R^2}\right)\Bigg|_{R/2}^r$$

$$Q_{en} = Q_1 + 4\pi\alpha\left(\frac{r^3}{3} - \frac{R^3}{24} - \frac{r^5}{5R^2} + \frac{R^3}{160}\right) = \frac{3}{32}\pi\alpha R^3 + 4\pi\alpha R^3\left(\frac{1}{3}\frac{r^3}{R^3} - \frac{1}{24} - \frac{1}{5}\frac{r^5}{R^5} + \frac{1}{160}\right)$$

$$Q_{en} = 4\pi\alpha R^3\left(\frac{3}{128} + \frac{1}{3}\left(\frac{r}{R}\right)^3 - \frac{1}{5}\left(\frac{r}{R}\right)^5 - \frac{17}{480}\right) = 4\pi R^3\left(\frac{480Q}{233\pi R^3}\right)\left(\frac{1}{3}\left(\frac{r}{R}\right)^3 - \frac{1}{5}\left(\frac{r}{R}\right)^5 - \frac{23}{1920}\right)$$

$$E_2 = \frac{1}{4\pi\epsilon_0}\frac{Q_{en}}{r^2} = \frac{4\pi R^3}{4\pi\epsilon_0 r^2}\left(\frac{480Q}{233\pi R^3}\right)\left(\frac{1}{3}\left(\frac{r}{R}\right)^3 - \frac{1}{5}\left(\frac{r}{R}\right)^5 - \frac{23}{1920}\right)$$

$$\boxed{E_2 = \frac{480Q}{233\pi\epsilon_0 r^2}\left(\frac{1}{3}\left(\frac{r}{R}\right)^3 - \frac{1}{5}\left(\frac{r}{R}\right)^5 - \frac{23}{1920}\right)}$$

For the region where $r \geq R$
The enclosed charge is simply equal to the total charge Q.

$$\boxed{E_3 = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{r^2}}$$

Part (c)

The fraction of enclosed charge is simply the ratio between Q_2 and Q.

$$\frac{Q_2}{Q} = \frac{47}{120}\pi\alpha R^3 \div \frac{233}{480}\pi\alpha R^3 = \boxed{0.807}$$

Part (d)

Using either E_2 or E_3 for $r = R/2$

$$\boxed{E = \frac{180}{233}\frac{Q}{4\pi\epsilon_0 R^2}}$$

Part (e)

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}}$$

The condition for simple harmonic motion is to have $F \propto r$ which is not the case here as $F \propto \dfrac{1}{r^2}$

Problem 2

University Physics, problem 48 (exam problem)

A small conducting spherical shell with inner radius a and outer radius b is concentric with a larger conducting spherical shell with inner radius c and outer radius d. The inner shell has total charge $+2q$ the outer shell has charge $-2q$.

a. Calculate the electric field (magnitude and direction) in terms of q and the distance r from the common center of the two shells for

   i. $r < a$
   ii. $a < r < b$
   iii. $b < r < c$
   iv. $c < r < d$
   v. $r > d$

b. What is the total charge on the

   i. inner surface of the small shell
   ii. outer surface of the small shell
   iii. inner surface of the large shell
   iv. outer surface of the large shell

Solution

Part (a)

i. $E_1 = 0$; charge enclosed is 0.

ii. $E_2 = 0$; inside a conductor.

iii. $E_3 = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2} = \frac{1}{2\pi\epsilon_0}\frac{q}{r^2}$

iv. $E_4 = 0$; inside a conductor.

v. $E_5 = 0$; charge enclosed is $-2q + 2q = 0$.

Part (b)

To solve this part, we take a Gaussian surface in the regions where we want to find the charge.

i. We take a Gaussian surface with radius r where $a < r < b$. Electric field is zero thus the charge enclosed is zero.

ii. We take a Gaussian surface with radius r where $b < r < c$. Electric field in this region is the same as that of a point charge $+2q$, the charge on the surface is $+2q$

iii. We take a Gaussian surface with radius r where $c < r < d$. Electric field in this region is zero, which means that the enclosed charge is zero. The enclose charge $Q_{en} = +2q + q_{\text{inner surface}} = 0 \implies q_{\text{inner surface}} = -2q$

iv. This spherical shell overall has a charge of $-2q$ which is already present on the inner surface, thus the charge on the outer surface is zero.